

THE IONIZATION-RATE CONSTANT AT HIGH TEMPERATURES HIGH ELECTRON CONCENTRATIONS

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The electron distribution function and the rate constant for ionization of atoms by electron impacts have been calculated as they apply to the conditions that are characteristic of a shock wave — namely, the energy distribution of the electrons and the ionization-rate constants are determined as functions of the temperature of the heavy particles. The energy dependence of the effective cross section for the excitation of an atom by electron impact is assumed to be linear. Equations of the Fokker-Planck type are used in the solution of the problem, and the range of temperatures and concentrations in which the deviation of the distribution from Maxwellian leads to a substantial change of the ionization-rate constant is determined.

The resultant ionization rate due to electron impact, which includes multistep transitions through the excited states of the atom, is proportional at high temperatures to the number of electrons whose energy is sufficient to excite the first level of the atom. In the process of such an excitation the electron loses an energy $\Delta\varepsilon \geq E_1$ (E_1 is the energy of the first excited state, measured from the ground level). If the average electron energy is several times as small as $\Delta\varepsilon$, then in the overwhelming majority of cases the electron no longer has an energy reserve sufficient to repeat such a collision after a collision that ends in the excitation of an atom. Thus, ionization of collisions leads to a reduction of the density of electrons $n(\varepsilon > E_1)$ capable of exciting an atom (i.e., the number $n(\varepsilon > E_1)$ that determines the ionization rate is itself dependent on this rate).

The problem of determining the ionization-rate constant can be reduced to calculating the electron distribution function. This distribution function differs from the equilibrium distribution during the process of ionization in a shock wave and is determined by the balance between the departure of electrons from the high-energy region during inelastic collisions and the replenishment of these losses due to elastic collisions of electrons with electrons, atoms, and ions.

A simple approximate method of calculating the ionization-rate constant of a gas in a shock wave has been developed by Petschek and Byron [1]. In accordance with [1] the electron distribution function is assumed to be Maxwellian at all energies, and its only difference from the equilibrium distribution function resides in the fact that the electron temperature T_e is not equal to the temperature T of the heavy particles during the ionization process. The electron temperature is calculated by means of the L. D. Landau relaxation equation [2] with allowance for the expenditure of electron energy for ionization. The temperature T_e does not differ from T in order of magnitude ($T_e < T$) even for a pronounced deviation of the gas composition from the equilibrium composition; however, the ionization-rate constant, which depends exponentially on the electron temperature, is much smaller than its equilibrium value corresponding to equality of the temperatures $T_e = T$. Further on, the problem of the deviation of the electron distribution function from the equilibrium distribution function due to various inelastic processes and of the corresponding change of the ionization-rate constant or of the excitation of atoms has been considered in a number of papers (see, for example, [3-7]). The present paper is a direct continuation of an investigation of the ionization of a monatomic gas in a shock wave begun by Petschek and Byron. The difference between the present statement

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of the problem and [4] resides in the fact that the electron distribution function is not assumed to be Maxwellian.

1. Statement of the Problem

Let us consider an optically dense gas having a constant atomic temperature T far from ionization equilibrium where recombination may be neglected. We shall consider the temperature to be sufficiently high so that the ionization process does not have a diffusion character and is determined by the kinetics of the transition of an atom from the ground state to the first excited state. Direct transitions from the ground state to higher levels are neglected as a consequence of the smallness of the corresponding cross sections. The scheme considered by the ionization process is realized in the case when the quasistationary population n_1 established for the first excited state during ionization is much lower than its equilibrium† value n_1^* .

The desired temperature boundary above which

$$n_1 \ll n_1^* \quad (1.1)$$

is determined from the balance of electrons situated at the first excited level in the quasistationary approximation:

$$dn_1/dt = 0 \quad (1.2)$$

Using (1.2), it may be shown that the population n_1 is determined by the following relationship for the condition (1.1):

$$n_1 = n_1^* \left(\frac{\sigma_{1e}}{\sigma_{10}} \exp \frac{E_1 - E}{T_e} + \frac{\sigma_{12}}{\sigma_{10}} \exp \frac{E_1 - E_2}{T_e} \right)^{-1}$$

where σ_{1e} is the cross section for the transition from the first excited state to the continuous spectrum; σ_{ik} is the cross section of the transition between the i and k bound states. Consequently, in order for (1.1) to be fulfilled it is, in any case, sufficient to require that

$$\exp(-I_1/T_e)(\sigma_{1e} + \sigma_{12})/\sigma_{10} \gg 1$$

where I_1 is the potential for ionization from the first excited state of the atoms. Typical values of the ratio $(\sigma_{1e} + \sigma_{12})/\sigma_{10}$ have an order of magnitude [8-10] equal to 10^2 . From this we obtain $5I_e > I_1$ for the temperature boundary. For atoms for which, as in hydrogen, the ratio between the potentials for ionization from the first and ground states is $I_1/I = 1/4$ the inequality may be written in the form

$$20T_e > I \quad (1.3)$$

The kinetic equation for the electron distribution function $f(t, \epsilon)$ has the form

$$f' = I_{ee} + I_{ea} + I_{ei} + I_i \quad (1.4)$$

In the right side of Eq. (1.4) the symbols I_{ee} , I_{ea} , and I_{ei} denote the corresponding integrals for collisions of electrons with electrons, atoms, and ions. The term I_i takes ionization into account.

We shall assume that the energy is substantially lower than during electron-atom elastic collisions. This imposes a specific condition on the electron concentration $c = n_e/n_a$ (n_e and n_a are the numbers of electrons and atoms per unit volume). If $\tau_{ee}(\epsilon)$ and $\tau_{ea}(\epsilon)$ respectively denote the electron-electron and electron-atom relaxation times, then the conditions imposed on the concentration can be written in the form

$$\tau_{ee} \ll \tau_{ea} \quad (1.5)$$

†We have in mind equilibrium relative to the ground state of the atom.

The relaxation time is

$$\tau(\varepsilon) \sim T^2 / \langle \partial (\Delta E)^2 / \partial t \rangle$$

where $\langle \sigma (\Delta E)^2 / \partial t \rangle$ is the mean-square of the energy change of an electron per unit time for the given type of collisions in a medium that is in thermodynamic equilibrium. For electron-electron collisions

$$\langle \partial (\Delta E_{ee})^2 / \partial t \rangle = 2^{1/2} \pi e^4 Q n_e T_e / (m\varepsilon)^{1/2} \quad (1.6)$$

where m is the electron mass and Q is the Coulomb logarithm. For electron-atom elastic collisions having a scattering cross section σ_{ea} we have

$$\langle \partial (\Delta E_{ea})^2 / \partial t \rangle = 2^{1/2} M^{-1} m^{1/2} \sigma_{ea} n_a T_e^{3/2}$$

where M is the mass of an atom. The condition (1.5) yields

$$c \gg m\varepsilon^2 / M T_e^2 Q \quad (1.7)$$

for the energy range $\varepsilon > E_1$ with allowance for the fact that $e^4 / \sigma_{ea} \sim \bar{I}^2$.

Besides the quantities τ_{ee} and τ_{ea} there is still another time scale in the problem – the characteristic ionizational relaxation time $\tau_i \sim (K n_a)^{-1}$, where K is the ionization-rate constant. The case when τ_i satisfies the inequality

$$\tau_i \gg \tau_{ee} \quad (1.8)$$

is of greatest practical interest.

Equations (1.5) and (1.8) allow considerable simplification of the kinetic equation (1.4). In the interval $0 < \varepsilon < E_1$ there are no elastic collisions of electrons with unexcited atoms. Therefore, during a time of the order of τ_{ee} a Maxwell distribution having the temperature T_e is established in this energy range; this distribution is somewhat violated as a consequence of collisions with heavy particles and the diffusion flux of electron energy in the interval $\varepsilon > E_1$ where there are large excitation losses, and also as a consequence of inelastic collisions with atoms that are in higher excited states. Each of these processes disrupts the Maxwell distribution of the electrons only insignificantly.

The first is ineffective as a consequence of the inequality (1.5) and

$$\tau_{ee} \sim \tau_{ei} m / M \ll \tau_{ei} \quad (1.9)$$

Here τ_{ei} is the relaxation time for elastic collisions between electrons and ions.

The second source of violation of the Maxwell distribution is small, because in the interval $\varepsilon > E_1$ (in the investigated temperature range) an exponentially small part of the overall number of electrons is in a gas that is in thermodynamic equilibrium and consequently the energy flux in the absence of equilibrium is exponentially small.

The third process is insubstantial because the number of electrons situated in higher excited states is exponentially small in comparison with the overall number of electrons. Thus, in the interval $\varepsilon < E_1$ there is a Maxwell distribution, and the problem can be reduced to calculating the electron temperature that may be found from the relaxation equation for a two-temperature gas with ionization energy losses.

It may be shown that in order to fulfill the inequality (1.8) the temperature T_e must not be too high; namely, $T_e < I\alpha^{-1}$, where $\alpha \sim 5-6$. From what is said further on it will become evident that this condition, which limits the temperature from above, is always fulfilled for the quasistationary stage of ionization in the shock wave.

In order to write the relaxation equation in closed form it is necessary first to find the electron distribution function at high energies $\varepsilon > E_1$ on which the ionization rate depends. The deviation of the electron distribution function from Maxwellian in the interval $\varepsilon > E_1$ is considerably larger than it is at low energies.

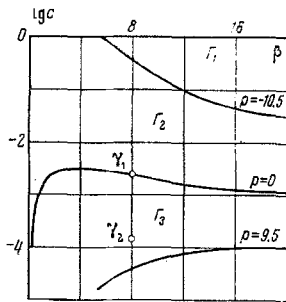


Fig. 1

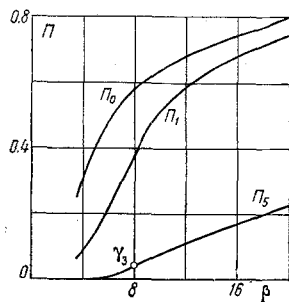


Fig. 2

This is related to the fact that for $\varepsilon > E_1$ an intensive energy drain occurs. The cross sections for the excitation of atoms by electrons are close to the gas-kinetic cross sections, and for each such collision an energy $\Delta\varepsilon \gg T_e$ is lost. The mechanism that compensates these losses — elastic collisions of electrons with electrons and with heavy particles — is characterized by effective cross sections that are also close to the gas-kinetic cross sections for electron-electron collisions and are $\sim m/M$ times as small for collisions with heavy particles. Moreover, the effective Coulomb cross sections for $\varepsilon > E_1$ are considerably smaller than for $\varepsilon \sim T_e$.

As a consequence of the pronounced violation of the Maxwell distribution of the electrons for $\varepsilon > E_1$, the collision integrals I_{ee} , I_{ea} , and I_{ei} are related to each other in inverse proportion to the corresponding relaxation times. Consequently, in accordance with (1.5) and (1.9) the integrals for collisions with heavy particles may be neglected at energies $\varepsilon > E_1$.

The restoration of equilibrium in the interval $\varepsilon > E_1$ takes place chiefly via collisions with electrons having an energy $\varepsilon < E_1$ (such electrons are the overwhelming majority), which have a Maxwell distribution and are characterized by the temperature T_e . This means that the restoration of the equilibrium distribution function may be treated as the relaxation in a medium having a temperature T_e . The collision integral I_{ee} may be represented in the form of a differential relationship of the Fokker-Planck type by expanding the distribution function in the magnitude of the energy transferred during a collision. Such an operation is legitimate, since the entire expansion is multiplied by the Rutherford cross section which has a sharp maximum in the range of small angles that are precisely the ones corresponding to small energy transfers. Taking account of all that has been said above, one may represent (1.4) in the form of an equation of the Fokker-Planck type with a source

$$f = \frac{1}{\varphi(\varepsilon)} \frac{\partial}{\partial \varepsilon} \left[\frac{\varphi(\varepsilon)}{2} \left\langle \frac{\partial}{\partial t} (\Delta E_{ee})^2 \right\rangle \left(\frac{\partial f}{\partial \varepsilon} + \frac{f}{T_e} \right) \right] - \sigma_i(\varepsilon) n_a \left(\frac{2\varepsilon}{m} \right)^{1/2} f \quad (1.10)$$

Here $\langle \partial(\Delta E_{ee})/\partial t \rangle$ is defined by Eq. (1.6); the density of the levels is $\varphi(\varepsilon) = 4\pi m^{-3/2} (2\varepsilon)^{1/2}$; σ_i is the cross section for the excitation of an atom from the ground state to the first excited state and is interpolated by a linear function of the energy [11, 12]

$$\sigma_i(\varepsilon) = \sigma \theta(\varepsilon - E_1)(\varepsilon - E_1)/E_1$$

Here $\theta(x)$ is the Heaviside function; σ depends on the specific type of atoms. For the majority of elements $\sigma \sim 10^{-16} \text{ cm}^2$.

Equation (1.10) permits further simplification as it applies to the problem considered. Let us introduce a distribution function that is normalized to one electron $f = n_e f_1$. Then the left side of Eq. (1.10) is written in the form

$$f = f_1 n_e + f_1 n_e \dot{}$$

In accordance with the definition of the ionization-rate constant K we have the following result far from ionizational equilibrium where the recombination process may be neglected:

$$n_e \dot{} = K n_e n_a \quad (1.11)$$

The ionization-rate constant is related to f_1 as follows:

$$K = \int_{E_1}^{\infty} \sigma_i \left(\frac{2\varepsilon}{m} \right)^{1/2} f_1 \varphi d\varepsilon \quad (1.12)$$

Comparing the quantity $f_1 n_e$ of the last term in the right side of Eq. (1.10) and assuming that in the temperature range considered the inequality

$$\int_{E_1}^{\infty} f_1(\varepsilon) \varphi(\varepsilon) d\varepsilon \ll 1$$

holds, we directly obtain

$$K n_e n_a f_1 \ll \sigma_i(\varepsilon) n_e n_a (2\varepsilon/m)^{1/2} f_1$$

for all $\varepsilon > E_1$ (with the exception of a very narrow energy range near the excitation threshold). Therefore the term $f_1 n_e$ in the kinetic equation (1.10) may be dropped. The term $f_1 n_e$ remaining in the left side of (1.10) vanishes after a time τ_{ee} has elapsed, and the entire subsequent ionization process takes place in a quasistationary operating mode which is described by the equation*

$$\frac{1}{\varphi(\varepsilon)} \frac{d}{d\varepsilon} \left[\frac{8\pi^2 Q n_e e^4 T_e}{m^2} \left(\frac{df_1}{d\varepsilon} + \frac{f_1}{T_e} \right) \right] - \sigma n_a \left(\frac{2\varepsilon}{m} \right)^{1/2} \frac{\varepsilon - E_1}{E_1} f_1 = 0 \quad (1.13)$$

2. The Solution of the Kinetic Equation

Let us write Eq. (1.13) in the dimensionless variables $x = \varepsilon/T_e$, $\beta = E_1/T_e$. Then instead of (1.13) we shall have

$$f_1''(x) + f_1'(x) - \kappa x(x - \beta) f_1(x) = 0, \quad x > \beta \\ \kappa = \sigma n_a T_e^3 / \pi Q e^4 n_e E_1 = \kappa_0 / c^3, \quad \kappa_0 = \sigma E_1^3 / \pi Q e^4 \quad (2.1)$$

The solution of Eq. (2.1) that satisfies the boundary condition at infinity $f_1 \rightarrow 0$, $x \rightarrow +\infty$, has the form

$$f_1 = B D_p(z) \exp -x/2, \quad z = (4\kappa)^{1/4} (x - \beta/2) \\ p = -\frac{1}{2} + \frac{\kappa \beta^2 - 1}{8 \sqrt{\kappa}}$$

Here $D_p(z)$ is the function for a parabolic cylinder [13]. For the subsequent analysis it is convenient to introduce the dimensionless quantities

$$\omega = (4\kappa)^{1/4}, \quad a = 1/2 \beta \omega, \quad p = -1/2 + 1/4 (a^2 - \omega^{-2})$$

In the energy range $0 < \varepsilon < E_1$ the distribution function differs considerably from Maxwellian. If we neglect this difference and choose it in the form

$$f_1 = (m/2\pi T_e)^{3/2} e^{-x}$$

it follows that, as will be shown below, the derivatives to the right and left for $x = \beta$ are practically equal. If this fact is taken into account, then the constant B is found immediately, and the distribution function in the range of high energies is finally written as

$$f_1 = \left(\frac{m}{2\pi T_e} \right)^{3/2} \frac{D_p(\omega(x - \beta/2))}{D_p(a)} \exp \frac{-(x + \beta)}{2} \quad (2.2)$$

Let us make several comments with respect to the behavior of the function $D_p(z)$. For fairly small ω (i.e., for fairly large concentrations and low temperatures (large β)) the index p takes large negative values. To the extent that the concentration decreases and the temperature increases, the index decreases in modulus; for certain values of c and β it passes through zero, and then it returns to values ~ 10 . A further increase is restricted by the condition (1.7).

*Strictly speaking, during a time $t > \tau_{ee}$ an operating mode is established in which the difference between the two collision integrals I_{ee} and I_i is not zero but much less than either one of them.

In the range of low temperatures and high concentrations the distorting effect of ionization is present to the least extent, and consequently in this range the form of the distribution function (2.2) is closest to Maxwellian. The limit $\omega \rightarrow 0$ corresponds to $z \rightarrow 0$, $p \rightarrow -\infty$, $z^2 p \rightarrow \text{const}$ being valid. Making the transition in the limit in (2.2), we find (see Appendix) that

$$f_1(x) \rightarrow (m/2\pi T_e)^{3/2} e^{-x} \quad (2.3)$$

3. Calculating the Ionization-Rate Constant and Electron Temperature

The ionization-rate constant K is determined by the integral (1.12). This integral is calculated most simply if one uses Eq. (1.13) after multiplying it by $\varphi(\epsilon)$ and integrating from E_1 to $+\infty$. This yields

$$K = - \frac{8\pi^2 Q e^4 T_e^c}{m^2} \left(\frac{df_1}{d\epsilon} + \frac{f_1}{T_e} \right) \Big|_{\epsilon=E_1}$$

Substituting the expression for the distribution function (2.2) into the expression above, we obtain

$$K = - \frac{\sqrt{2\pi} Q e^4 c e^{-\beta}}{\sqrt{m T_e}^{3/2}} \left(1 + 2\omega \frac{D_p'(a)}{D_p(a)} \right) \quad (3.1)$$

In order to calculate (3.1) let us consider the plane $c\beta$ and draw lines having equal indices p on it (i.e., solutions of the equation $p = \text{const}$).

The point γ_1 on the graph (Fig. 1) corresponds to the equilibrium concentration for He at $T_e = 2.46$ eV and an atomic density $2.7 \cdot 10^{19} \text{ cm}^3$. The points lying on the vertical straight line below the point γ_1 characterize a gas state in which the concentration of charged particles is lower than the equilibrium value.

The line $p = 0$ divides the entire plane into two parts. In the upper part the index $p < 0$. It increases in modulus when the transition is made from line to line upward. In the right upper corner of the plane the distribution function goes over to a Maxwellian distribution function. This corresponds to the limit (2.3). It is therefore clear that in the domain I_1 the function $D_p(z)$ is defined by the asymptotic behavior obtained for the conditions $z \rightarrow 0$, $p \rightarrow \infty$ in such a way that $z^2 p = \text{const}$. This asymptotic behavior is found in the Appendix. The transitional domain I_2 from the asymptotic behavior to the values of the function $D_p(z)$ for $p > 0$ is fairly narrow (the index p varies from 0 to ~ -10 in it).

In the domain I_3 of the plane $c\beta$ Eq. (3.1) is calculated simply for integer p , since the function $D_p(z)$ in this case can be reduced to Hermite polynomials H_p :

$$D_p(z) = 2^{-1/2 p} \exp(-z^2/4) H_p(z/\sqrt{2})$$

In order to calculate the ionization-rate constant for fractional values of p it is sufficient to use linear interpolation with respect to p .

Let us now go over to determining the temperature of the Maxwell distribution of the electrons at energies $\epsilon < E_1$. Let us write the energy-balance equation with allowance for electron-atom and electron-ion elastic collisions and inelastic collisions of electrons with atoms ending with ionization:

$$\frac{d}{dt} \left(\frac{3}{2} T_e n_e \right) = \int_0^\infty I_{ea} \epsilon \varphi d\epsilon + \int_0^\infty I_{ei} \epsilon \varphi d\epsilon + \int_0^\infty I_i \epsilon \varphi d\epsilon \quad (3.2)$$

The rate of change of the electron energy with time appears in the left side of (3.2). This term may be represented in such a form because the overwhelming majority of the electrons has a Maxwellian distribution function with a temperature T_e . Taking account of the fact that the atoms and ions have a Maxwell distribution with a temperature T , Eq. (3.2) may be given the form

$$T_e \dot{=} - (I + T_e) n_a K - \frac{32\sigma_{ea} n_a \sqrt{m T_e}}{3 \sqrt{2\pi} M} (T_e - T) - \frac{4 \sqrt{2\pi} e^4 Q n_e \sqrt{m}}{3 M T_e^{3/2}} (T_e - T) \quad (3.3)$$

Here we have taken into account the fact that the derivative dn_e/dt is defined by (1.11). We shall seek the solution of Eq. (3.3) in the quasistationary approximation (i.e., we neglect the dependence of T_e on time); then from (3.3) we immediately obtain

$$T = T_e \left\{ 1 + \left(1 + \frac{I}{T_e} \right) \frac{3}{4} \frac{M}{m} \left(1 + \frac{4\sigma_{ea} T_e^2}{\pi Q e^4 c} \right)^{-1} \left(-1 + 2\omega \left| \frac{D_p'(a)}{D_p(a)} \right| \right) e^{-\beta} \right\} \quad (3.4)$$

The function T_e depends approximately logarithmically on T and on other parameters that are included in the transcendental equation (3.4). Therefore for any reasonable values of the parameters the electron temperature in the quasistationary state of ionization is bounded from above. Thus, for example, if $T \sim I$, then $T_e \sim I/5$.

4. Analysis of the Results

Consideration of the deviation of the distribution function of the electrons from Maxwellian leads to a reduction of the ionization-rate constant for a stipulated value of T_e . If the electron distribution function at all energies is characterized by the temperature T_e , then, as is well known,

$$K(T_e) = \frac{4\sigma \sqrt{T_e}}{\sqrt{2\pi m}} \left(1 + \frac{2}{\beta} \right) e^{-\beta}$$

Let us introduce

$$K = K(T_e) \Pi$$

Since in the domain Γ_1 the "tail" of the distribution is slightly distorted, the function Π there differs only slightly from unity. Actually, from (3.1) we obtain an expression for Π which may be expanded in powers of ω by using the smallness of this quantity in the domain Γ_1 :

$$\Pi = 1 - \omega^2 + O(\omega^4)$$

The greatest difference between Π and unity should be expected in the domain Γ_3 , because in this domain the distribution function is greatly distorted. Let us perform the calculations for integer $p = n$ and assign the appropriate number to all of the values. From the equations $p = n$ we determine the dependence of the concentrations on temperature:

$$c_n(\beta) = \kappa_0 [\beta + 8(n + 1/2)(1 + 16\beta^{-2}(n + 1/2)^2)^{1/2} + 32\beta^{-1}(n + 1/2)^2]^{-1}$$

The functions Π_n may be obtained from the general result (3.1). The results of the calculations are conveniently represented in the form of a graph.

The state of He gas described by the point γ_2 in Fig. 1 corresponds to point γ_3 in Fig. 2. From the graphs it is evident that in the domain Γ_3 consideration of the distortion of the tail of the distribution leads to a substantial change of the ionization-rate constant. As the concentration decreases, this change increases. The case of low concentrations that do not correspond to the inequality (1.7) (i.e., concentrations for which elastic collisions of electrons with atoms in a high-energy range become substantial) differs in a number of features. This case is not considered in the present paper.

It remains for us to show the validity of the matching of the distribution function (2.2). The derivative on the left side is obtained by differentiating the Maxwellian distribution and is equal to $-T_e (m/2\pi T_e)^{3/2} e^{-\beta}$ at the point $x = \beta$. The derivative on the right side is obtained from Eq. (2.2), which is very close to (2.3) in the domain Γ_1 . The greatest difference between the derivatives is attained in the domain Γ_3 , where the distorting effects of the ionization become substantial. Nevertheless, the possibility of matching is still preserved. For example, for $n = 0$ the derivative on the right is equal to

$$-T_e \left(\frac{m}{2\pi T_e} \right)^{3/2} e^{-\beta} \left[\frac{1}{2} + \frac{1}{2} \left(1 + \frac{4}{\beta} \left(1 + \frac{4}{\beta} \left(1 + \frac{4}{\beta^2} \right)^{1/2} + \frac{8}{\beta^3} \right)^{1/2} \right) \right] \quad (4.1)$$

The expression in the square brackets of (4.1) is practically no different from unity wherever the inequality (1.3) is fulfilled.

APPENDIX

Let us find the asymptotic form of the function $D_p(z)$ for the condition $z \rightarrow 0$, $p \rightarrow -\infty$, $z^2 p \rightarrow \text{const}$. Let us use the integral representation for the function $D_p(z)$ in the case of negative p :

$$D_p(z) = \frac{1}{\Gamma(-p)} \int_0^\infty \exp\left(-\frac{z^2}{4} - zx - \frac{x^2}{2}\right) \frac{dx}{x^{p+1}} \quad (1)$$

We calculate this integral by the method of steepest descent. For the inflection point x_0 we obtain

$$2x_0 = (z^2 + 4k)^{1/2} - z, \quad k = -(p + 1)$$

Now (1) takes the form

$$\frac{1}{\Gamma(-p)} \exp\left(-zx_0 - \frac{x_0^2}{2} + k \ln x_0 - \frac{z^2}{4}\right) \int_0^\infty \exp\left[\left(-1 - \frac{k}{x_0^2}\right) \frac{(x - x_0)^2}{2}\right] \quad (2)$$

Equation (2) is equal to (1) with high accuracy if

$$x_0 \gg (1 + k/x_0^2)^{-1/2} \quad (3)$$

On the cz plane the inequality (3) defines the domain of applicability of the desired asymptotic behavior in the problem given. A numerical analysis of it shows that the asymptotic behavior is a good representation of the function $D_p(z)$ for $p \lesssim -10$. When the condition (3) is fulfilled the integral (2) is easily calculated. The result has the following form with an accuracy of up to the term z^2/\sqrt{k} .

$$\frac{\sqrt{\pi}}{\Gamma(-p)} \left(1 - \frac{z}{4\sqrt{k}}\right) \exp\left[\frac{k}{2}(\ln k - 1)\right] \exp\left(-z\sqrt{\frac{k}{2}} - \frac{z^2}{8}\right)$$

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